Chapter 11
Rotational Dynamics and Static Equilibrium
11-1 Torque

From experience, we know that the same force will be much more effective at rotating an object such as a nut or a door if our hand is not too close to the axis.

This is why we have long-handled wrenches, and why doorknobs are not next to hinges.
11-1 Torque

We define a quantity called torque:

**Definition of Torque, \( \tau \), for a Tangential Force**

\[ \tau = rF \]

SI unit: \( \text{N} \cdot \text{m} \)

The torque increases as the force increases, and also as the distance increases.
11-1 Torque

Only the tangential component of force causes a torque:

Zero torque

Torque = \( r(F \sin \theta) \)

\[ F \cos \theta \]

\[ F \sin \theta \]
11-1 Torque

This leads to a more general definition of torque:

**General Definition of Torque, \( \tau \)**

\[
\tau = r(F \sin \theta)
\]

SI units: N \( \cdot \) m
11-1 Torque

If the torque causes a counterclockwise angular acceleration, it is positive; if it causes a clockwise angular acceleration, it is negative.
Newton’s second law: \[ a = \frac{F}{m} \]

If we consider a mass \( m \) rotating around an axis a distance \( r \) away, we can reformat Newton’s second law to read:

\[ \alpha = \frac{rF}{mr^2} = \frac{\tau}{I} \]

Or equivalently,

**Newton's Second Law for Rotational Motion**

\[ \tau = I\alpha \]
11-2 Torque and Angular Acceleration

Once again, we have analogies between linear and angular motion:

<table>
<thead>
<tr>
<th>Linear Quantity</th>
<th>Angular Quantity</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m$</td>
<td>$I$</td>
</tr>
<tr>
<td>$a$</td>
<td>$\alpha$</td>
</tr>
<tr>
<td>$F$</td>
<td>$\tau$</td>
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</tbody>
</table>
92. **BIO Roller Pigeons** Pigeons are bred to display a number of interesting characteristics. One breed of pigeon, the “roller,” is remarkable for the fact that it does a number of backward somersaults as it drops straight down toward the ground. Suppose a roller pigeon drops from rest and free falls downward for a distance of 14 m. If the pigeon somersaults at the rate of 12 rad/s, how many revolutions has it completed by the end of its fall?
\[ y = y_0 + v_{0y}t - \frac{1}{2}gt^2 \]
\[ 0 = y_0 + 0 - \frac{1}{2}gt^2 \]
\[ t = \sqrt{2y_0/g} \]

\[ \Delta \theta = \omega_{av} \Delta t = \omega_{av} \sqrt{2y_0/g} \]
\[ = \left( \frac{12 \text{ rad}}{s} \right) \left( \frac{1 \text{ rev}}{2\pi \text{ rad}} \right) \sqrt{\frac{2(14 \text{ m})}{9.81 \text{ m/s}^2}} = 3.2 \text{ rev} \]
14. **IP** A wheel on a game show is given an initial angular speed of 1.22 rad/s. It comes to rest after rotating through 0.75 of a turn. (a) Find the average torque exerted on the wheel given that it is a disk of radius 0.71 m and mass 6.4 kg. (b) If the mass of the wheel is doubled and its radius is halved, will the angle through which it rotates before coming to rest increase, decrease, or stay the same? Explain. (Assume that the average torque exerted on the wheel is unchanged.)
\[ I = \frac{1}{2} MR^2 = \frac{1}{2} (6.4 \text{ kg})(0.71 \text{ m})^2 = 1.6 \text{ kg} \cdot \text{m}^2 \]

\[ \alpha = \frac{\omega^2 - \omega_0^2}{2\Delta \theta} = \frac{0^2 - (1.22 \text{ rad/s})^2}{2(0.75 \text{ rev} \times 2\pi \text{ rad/rev})} = -0.158 \text{ rad/s}^2 \]

\[ \tau = I \alpha = (1.6 \text{ kg} \cdot \text{m}^2)(-0.158 \text{ rad/s}^2) = -0.25 \text{ N} \cdot \text{m} \]
17. **CE** A motorcycle accelerates from rest, and both the front and rear tires roll without slipping. (a) Is the force exerted by the ground on the rear tire in the forward or in the backward direction? Explain. (b) Is the force exerted by the ground on the front tire in the forward or in the backward direction? Explain. (c) If the moment of inertia of the front tire is increased, will the motorcycle’s acceleration increase, decrease, or stay the same? Explain.
Torque on wheel by motor

Static friction from ground to oppose rotation and make wheel roll. Large, to make entire motorcycle move forward.

Static friction from ground to create wheel rotation and make wheel roll. Smaller, because only front wheel inertia to overcome.
11-3 Zero Torque and Static Equilibrium

Static equilibrium occurs when an object is at rest – neither rotating nor translating.

**Conditions for Static Equilibrium**
For an extended object to be in static equilibrium, the following two conditions must be met:

(i) The net force acting on the object must be zero,

\[ \sum F_x = 0, \quad \sum F_y = 0 \quad 11-5 \]

(ii) The net torque acting on the object must be zero,

\[ \sum \tau = 0 \quad 11-6 \]
11-3 Zero Torque and Static Equilibrium

When forces have both vertical and horizontal components, in order to be in equilibrium an object must have no net torque, and no net force in either the $x$- or $y$-direction.
11-3 Zero Torque and Static Equilibrium

If the net torque is zero, it doesn’t matter which axis we consider rotation to be around; we are free to choose the one that makes our calculations easiest.
25. **IP BIO** Referring to the person holding a baseball in Problem 5, suppose the biceps exert just enough upward force to keep the system in static equilibrium. (a) Is the force exerted by the biceps more than, less than, or equal to the combined weight of the forearm, hand, and baseball? Explain. (b) Determine the force exerted by the biceps.
\[ \tau_{\text{biceps}} = r_{\perp, \text{biceps}} F \]
\[ \tau_{\text{forearm}} = r_{\perp} mg = (0.170 \text{ m})(1.20 \text{ kg})(9.81 \text{ m/s}^2) = -2.001 \text{ N} \cdot \text{m} \]
\[ \tau_{\text{ball}} = r_{\perp} W_{\text{ball}} = (0.340 \text{ m})(1.42 \text{ N}) = -0.483 \text{ N} \cdot \text{m} \]
\[ \sum \tau = \tau_{\text{biceps}} + \tau_{\text{forearm}} + \tau_{\text{ball}} = 0 \]
\[ = r_{\perp, \text{biceps}} F - 2.001 - 0.483 \text{ N} \cdot \text{m} = 0 \]
\[ F = \frac{2.484 \text{ N} \cdot \text{m}}{0.0275 \text{ m}} = 90.3 \text{ N} \]
If an extended object is to be balanced, it must be supported through its center of mass.
This fact can be used to find the center of mass of an object – suspend it from different axes and trace a vertical line. The center of mass is where the lines meet.
44. **Maximum Overhang** Three identical, uniform books of length $L$ are stacked one on top the other. Find the maximum overhang distance $d$ in Figure 11-32 such that the books do not fall over.
Solution: 1. The center of mass of book 3 needs to be above the right end of book 2:

\[ d_2 = \frac{L}{2} \]

center of mass of book 2 is located at \( \frac{L}{2} + \frac{L}{2} = L \) from the right edge of book 1.
The center of mass of books 3 and 2 needs to be above the right end of book 1

\[ d_1 = X_{cm,32} = \frac{m\left(\frac{L}{2}\right) + m(L)}{2m} = \frac{3}{4}L \]

The result of step 3 means that the center of mass of book 1 is located at \( 3L \frac{4}{4} + \frac{L}{2} = \frac{5L}{4} \). The center of mass of books 3, 2, and 1 needs to be above the right end of the table

\[ d = X_{cm,321} = \frac{m\left(\frac{L}{2}\right) + m(L) + m\left(\frac{5L}{4}\right)}{3m} = \frac{11}{12}L \]
**11-5 Dynamic Applications of Torque**

When dealing with systems that have both rotating parts and translating parts, we must be careful to account for all forces and torques correctly.

(a) Physical picture  
(b) Free-body diagram for mass  
(c) Free-body diagram for pulley
46. A 2.85-kg bucket is attached to a disk-shaped pulley of radius 0.121 m and mass 0.742 kg. If the bucket is allowed to fall, (a) what is its linear acceleration? (b) What is the angular acceleration of the pulley? (c) How far does the bucket drop in 1.50 s?
\[ \sum \tau = rT = I \alpha \]

\[ T = \frac{I \alpha}{r} = \left( \frac{1}{2} MR^2 \right) \left( \frac{a}{R} \right) = \frac{1}{2} Ma \]

\[ \sum F_y = -T + mg = ma \]

\[ -\left( \frac{1}{2} Ma \right) + mg = ma \]

\[ mg = \left( m + \frac{1}{2} M \right) a \]

\[ a = \left( \frac{m}{m + \frac{1}{2} M} \right) g = \left[ \frac{2.85 \text{ kg}}{2.85 + \frac{1}{2} (0.742) \text{ kg}} \right] (9.81 \text{ m/s}^2) = 8.68 \text{ m/s}^2 \]

\[ \alpha = \frac{a}{R} = \frac{8.68 \text{ m/s}^2}{0.121 \text{ m}} = 71.7 \text{ rad/s}^2 \]

\[ \Delta y = 0 + \frac{1}{2} a_y t^2 = \frac{1}{2} \left( 8.68 \text{ m/s}^2 \right) (1.50 \text{ s})^2 = 9.77 \text{ m} \]
49. **IP** You pull downward with a force of 28 N on a rope that passes over a disk-shaped pulley of mass 1.2 kg and radius 0.075 m. The other end of the rope is attached to a 0.67-kg mass. (a) Is the tension in the rope the same on both sides of the pulley? If not, which side has the largest tension? (b) Find the tension in the rope on both sides of the pulley.
\[ \sum F_y = T_2 - Mg = Ma \]
\[ \sum \tau = r T_1 - r T_2 = I \alpha = \left( \frac{1}{2} mr^2 \right) \left( \frac{a}{r} \right) \Rightarrow a = \frac{2(T_1 - T_2)}{m} \]

\[
T_2 - Mg = M \left[ 2 \left( T_1 - T_2 \right)/m \right] \\
mT_2 - mMg = 2MT_1 - 2MT_2 \\
T_2 = \frac{M \left( 2T_1 + mg \right)}{2M + m} = \frac{(0.67 \text{ kg}) \left[ 2 \left( 28 \text{ N} \right) + (1.2 \text{ kg}) \left( 9.81 \text{ m/s}^2 \right) \right]}{2 \left( 0.67 \text{ kg} \right) + 1.2 \text{ kg}} = 18 \text{ N}
\]

The net force on the hanging mass is thus \( T_2 - Mg = 18 - 6.6 \text{ N} = 11.4 \text{ N} \), enough to accelerate it upward at \( 17 \text{ m/s}^2 \). The angular acceleration of the pulley is thus \( \alpha = \left( 17 \text{ m/s}^2 \right) \left( 0.075 \text{ m} \right) = 230 \text{ rad/s}^2 \)
11-6 Angular Momentum

Definition of the Angular Momentum, $L$

$$L = I \omega$$

SI unit: kg \cdot m^2/s

Using a bit of algebra, we find for a particle moving in a circle of radius $r$,

$$L = rmv = rp$$
11-6 Angular Momentum

For more general motion,
Angular Momentum, \( L \), for a Point Particle

\[
L = r p \sin \theta = r m v \sin \theta
\]

SI unit: kg \cdot m^2/s
11-6 Angular Momentum

Looking at the rate at which angular momentum changes,

\[ \frac{\Delta L}{\Delta t} = I \frac{\Delta \omega}{\Delta t} \]

\[ = I \alpha \]

**Newton's Second Law for Rotational Motion**

\[ \tau = I \alpha = \frac{\Delta L}{\Delta t} \]
58. **IP** Suppose jogger 3 in Figure 11–33 has a mass of 62.2 kg and a speed of 5.85 m/s. (a) Is the magnitude of the jogger’s angular momentum greater with respect to point A or point B? Explain. (b) Is the magnitude of the jogger’s angular momentum with respect to point B greater than, less than, or the same as it is with respect to the origin, O? Explain. (c) Calculate the magnitude of the jogger’s angular momentum with respect to points A, B, and O.
Picture the Problem: Jogger 3 runs in a straight line at constant speed in the manner indicated by the figure at right.

Strategy: Use $L = r \cdot m \cdot v$ (equation 11-12) to find the angular momentum.

Solution:
1. (a) The angular momentum increases with the perpendicular distance to the reference point. Since jogger 3's perpendicular distance to point A is zero, his angular momentum is zero with respect to point A. Therefore his angular momentum is greater with respect to point B.
2. (b) Jogger 3 has the same perpendicular distance to point B as he has to the origin, O. Therefore, his angular momentum with respect to B is the same as it is with respect to O.
3. (c) Apply equation 11-12 directly:
   
   \[ L_A = r_{1,A} mv = (0.00 \text{ m})(62.2 \text{ kg})(5.85 \text{ m/s}) = 0 \text{ kg} \cdot \text{m}^2/\text{s} \]
   
   \[ L_B = r_{1,B} mv = (7.00 \text{ m})(62.2 \text{ kg})(5.85 \text{ m/s}) = 2.55 \times 10^3 \text{ kg} \cdot \text{m}^2/\text{s} \]
   
   \[ L_O = r_{1,O} mv = (7.00 \text{ m})(62.2 \text{ kg})(5.85 \text{ m/s}) = 2.55 \times 10^3 \text{ kg} \cdot \text{m}^2/\text{s} \]

Insight: The angular momenta of all 3 joggers have the same sign (they are all clockwise). If you use the right hand rule introduced in section 11-9, the angular momentum vectors of each point into the page.
60. A windmill has an initial angular momentum of 8500 kg·m²/s. The wind picks up, and 5.86 s later the windmill’s angular momentum is 9700 kg·m²/s. What was the torque acting on the windmill, assuming it was constant during this time?
58. **Picture the Problem**: Jogger 3 runs in a straight line at constant speed in the manner indicated by the figure at right.

**Strategy**: Use $L = m r v$ (equation 11-12) to find the angular momentum.

**Solution**:

1. (a) The angular momentum increases with the perpendicular distance to the reference point. Since jogger 3’s perpendicular distance to point A is zero, his angular momentum is zero with respect to point A. Therefore, his angular momentum is greater with respect to point B.

2. (b) Jogger 3 has the same perpendicular distance to point B as he has to the origin, O. Therefore, his angular momentum with respect to B is the same as it is with respect to O.

3. (c) Apply equation 11-12 directly:

$$L = m r v$$

$$L_A = (62.2 \text{ kg})(0.00 \text{ m})(5.85 \text{ m/s}) = 0 \text{ kg m}^2/\text{s}$$

$$L_A = 0 \text{ kg m}^2/\text{s}$$

4. Apply equation 11-12 directly:

$$L_B = (62.2 \text{ kg})(7.00 \text{ m})(5.85 \text{ m/s}) = 2.55 \times 10^5 \text{ kg m}^2/\text{s}$$

$$L_B = 2.55 \times 10^5 \text{ kg m}^2/\text{s}$$

5. Apply equation 11-12 directly:

$$L_O = (62.2 \text{ kg})(7.00 \text{ m})(5.85 \text{ m/s}) = 2.55 \times 10^5 \text{ kg m}^2/\text{s}$$

$$L_O = 2.55 \times 10^5 \text{ kg m}^2/\text{s}$$

**Insight**: The angular momenta of all 3 joggers have the same sign (they are all clockwise). If you use the right-hand rule introduced in section 11-9, the angular momentum vectors of each point into the page.

59. **Picture the Problem**: The egg beater rotates about its axis with constant angular acceleration due to the applied torque.

**Strategy**: Use equation 11-14 to find the change in angular momentum due to the applied torque. Then use equation 11-11 to find the angular speed of the egg beater.

**Solution**:

1. (a) Solve equation 11-14 for $L$:

$$\tau = \frac{L_2 - L_1}{\Delta t} = \frac{9700 - 8500 \text{ kg m}^2/\text{s}}{5.86 \text{ s}} = 200 \text{ N m} = 0.20 \text{ kN m}$$

2. (b) Solve equation 11-11 for $\Omega$:

$$\Omega = \frac{L}{I}$$

$$\Omega = \frac{2550000 \text{ kg m}^2/\text{s}}{1200 \text{ kg m}^2} = 31 \text{ rad/s}$$

**Insight**: As long as the torque is applied to the egg beater, its angular speed and angular momentum will increase linearly with time.

60. **Picture the Problem**: The windmill rotates about its axis with constant angular acceleration due to the applied torque.

**Strategy**: Use equation 11-14 to find the torque required to change the angular momentum of the windmill by the specified amount during the given time interval.

**Solution**:

Apply equation 11-14 directly:

$$\tau = \frac{L_2 - L_1}{\Delta t} = \frac{9700 - 8500 \text{ kg} \cdot \text{m}^2/\text{s}}{5.86 \text{ s}} = 200 \text{ N m} = 0.20 \text{ kN m}$$

**Insight**: Note that the 1200 kg·m$^2$/s change in angular momentum limits the answer to only two significant figures.

When the wind is blowing at a constant speed, the net torque on the windmill is zero and it rotates at constant speed.

61. **Picture the Problem**: The gerbils remain stationary but the exercise wheel rotates with constant angular speed.

**Strategy**: Because the gerbils are running in place, their speed is zero relative to the laboratory frame of reference and they contribute no angular momentum. Use equation 11-11 together with the moment of inertia of the hoop-shaped wheel ($I = 2IMR^2$) and the fact that the gerbils run without slipping ($v = R\Omega$) to find the angular momentum of the wheel.

**Solution**:

Apply equation 11-11 directly:

$$L = I\Omega = (2IMR^2)(3.14 \text{ m/s}) = 2.6 \times 10^5 \text{ kg m}^2/\text{s}$$

**Insight**: If the gerbils were to suddenly stop and clutch the rim of the exercise wheel, they would rotate with the wheel in a vertical circle, and would contribute to the angular momentum. At the instant they grab on to the wheel, the angular momentum would remain the same but the moment of inertia would increase to $2I_m R^2$ and the linear speed of the wheel would decrease to 0.0062 m/s.
11-7 Conservation of Angular Momentum

If the net external torque on a system is zero, the angular momentum is conserved.

The most interesting consequences occur in systems that are able to change shape:
As the moment of inertia decreases, the angular speed increases, so the angular momentum does not change.

Angular momentum is also conserved in rotational collisions:
The Masseter Muscle

The masseter muscle, the principal muscle for chewing, is one of the strongest muscles for its size in the human body. It originates on the lower edge of the zygomatic arch (cheekbone) and inserts in the angle of the mandible. Referring to the lower diagram in Figure 11-39, where \( d = 7.60 \text{ cm} \) and \( D = 10.85 \text{ cm} \), (a) find the torque produced about the axis of rotation by the masseter muscle. The force exerted by the masseter muscle is \( F_M = 455 \text{ N} \). (b) Find the biting force, \( F_D \), exerted on the mandible by the upper teeth. Find (c) the horizontal and (d) the vertical component of the force \( F_j \) exerted on the mandible at the joint where it attaches to the skull. Assume that the mandible is in static equilibrium, and that upward is the positive vertical direction.
\[ \tau = r \times F = (D - d)(F_M \cos \theta) \]
\[ = (0.1085 - 0.0760 \text{ m})[(455 \text{ N}) \cos 26.0^\circ] = 13.3 \text{ N} \cdot \text{m} \]

\[ \tau = r \times F = DF_B \]
\[ F_B = \frac{\tau}{D} = \frac{13.3 \text{ N} \cdot \text{m}}{0.1085 \text{ m}} = 123 \text{ N} \]

(c) Set \( \sum F_x = 0 \) to find \( F_{Jx} \):
\[ \sum F_x = -F_{Mx} + F_{Jx} = 0 \]
\[ F_{Jx} = F_{Mx} = F_M \sin \theta = (455 \text{ N}) \sin 26.0^\circ = 199 \text{ N} \]

(d) Set \( \sum F_y = 0 \) to find \( F_{Jy} \):
\[ \sum F_y = -F_B + F_{My} + F_{Jy} = 0 \]
\[ F_{Jy} = F_B - F_{My} \]
\[ = F_B - F_M \cos \theta = 123 \text{ N} - (455 \text{ N}) \cos 26.0^\circ = -286 \text{ N} \]
97. **Balancing a T. rex** Paleontologists believe that *Tyrannosaurus rex* stood and walked with its spine almost horizontal, as indicated in Figure 11-41, and that its tail was held off the ground to balance its upper torso about the hip joint. Given that the total mass of *T. rex* was 5400 kg, and that the placement of the center of mass of the tail and the upper torso was as shown in Figure 11-41, find the mass of the tail required for balance.
\[ r_T m_T g - r_U m_U g = 0 \]
\[ r_T m_T - r_U (M - m_T) = 0 \]
\[ M = m_U + m_T \]
\[ m_T = \frac{r_U M}{r_T + r_U} = \frac{(1.4 \text{ m})(5400 \text{ kg})}{2.4 + 1.4 \text{ m}} = 2000 \text{ kg} = 2.0 \times 10^3 \text{ kg} \]
108. **IP** Suppose partial melting of the polar ice caps increases the moment of inertia of the Earth from $0.331 \, M_E R_E^2$ to $0.332 \, M_E R_E^2$. (a) Would the length of a day (the time required for the Earth to complete one revolution about its axis) increase or decrease? Explain. (b) Calculate the change in the length of a day. Give your answer in seconds.
\[ I_0 \omega_0 = I_1 \omega_1 \]

\[ I_1 \left( \frac{2\pi}{T_1} \right) = I_i \left( \frac{2\pi}{T_i} \right) \Rightarrow T_i = T_i \left( \frac{I_f}{I_i} \right) = T_i \left( \frac{I_f}{I_i} \right) \]

\[ \Delta T = T_i - T_i = \left( \frac{I_f}{I_i} - 1 \right) T_i = \left( \frac{0.332 M_E R_E^2}{0.331 M_E R_E^2} - 1 \right) \left( 86,400 \text{ s} \right) = 261 \text{ s} \]