1. Ideal gas of two-state atoms

Consider an ideal monoatomic gas made of $N$ atoms each of which has only 2 internal states: a ground state and an excited state with energy gap equal to $\Delta$. The gas is in a sealed container with no energy exchange with outside world. Initially, the gas is prepared in such a way that all the atoms are in their ground state internally, but the gas is in thermal equilibrium with respect to kinetic motion of the atoms, characterized by temperature $T_1$. After some time, however, due to collisions, the internal degree of freedom of the atoms is also excited and thermalized.

a) Find the temperature of the gas, $T_2$, after the internal degree of freedom thermalizes. Assume $\Delta \ll kT_1$ and calculate the difference $T_2 - T_1$ up to order $\Delta$. Does the temperature increase or decrease?

b) Find the change in the entropy of the gas, $S_2 - S_1$, after the complete thermalization. Assume $\Delta \ll kT_1$ and work up to order $\Delta$. Does the entropy increase or decrease?

Hint: The entropy of the same gas without the internal degree of freedom is:

$$S_{\text{kin}} = \frac{3}{2} kN \ln T + (T \text{ independent terms}).$$

2. van der Waals equation of state

Consider van der Waals equation of state:

$$p = \frac{n k T}{1 - b n} - a n^2;$$

where $p$ is the pressure, $n$ is the density and $T$ is the temperature. $a$ and $b$ are positive coefficients.

a) There are 2 different types of isotherms $p(n)$, depending on the value of $T$. For some values of $T$ the pressure is a monotonous function of the density, for other values it is not. Sketch these two types of the isotherms on the $p$ vs $n$ plot.

b) Now, on the same plot, sketch the curve where the derivative $(\partial p/\partial n)_T$ vanishes.

c) Shade the region of $p$ and $n$ where the system is thermodynamically unstable towards phase separation (not even metastable). Write the stability condition (inequality) you use.

d) Express the coordinates $p_c$, $n_c$ and $T_c$ of the critical point in terms of $a$ and $b$. 

3. Maxwell relations

Consider a rubber band of length $L$ which is being stretched by external force $f$.

a) Write down the thermodynamic identity (1st law of thermodynamics) relating change in the internal energy $dU$ to infinitesimal change in length $dL$, and to supplied heat $TdS$.

b) In one experiment the length of the band is fixed to $L = 1$ m and the temperature of the band $T = 300$ K is raised by a small amount $\Delta T = 3$ K. This causes the force needed to maintain the length of the band to increase by the amount $\Delta f = 1.2$ N. In another experiment, the band is stretched from $L$ to $L + \Delta L$ at constant temperature $T$. As a result the band exchanges heat with the environment. What is the amount of this heat for $\Delta L = 2$ cm? Is the heat released or absorbed by the band?

4. Cosmic microwave background

Cosmic microwave background (CMB, or relic) radiation is an isotropic radiation with a black body spectrum at temperature $T = 2.7$ K.

a) Find the density $n$ of the CMB photons. How many relic photons are there on average inside a volume of space $V = 1$ cm$^3$?

b) Find the rate at which a ball of radius $R = 1$ cm is struck by relic photons.

You may find useful the following combination of constants:

$$\frac{k}{hc} = 436.7 \text{ K}^{-1}\text{m}^{-1}$$

as well as this integral

$$\int_0^\infty dx \frac{x^2}{e^x - 1} = 2\zeta(3) = 2.404\ldots$$
5. Ultrarelativistic Fermi gas at $T = 0$

Matter inside a star can be compressed to such an extent that the Fermi energy of the electrons becomes much larger than their rest energy. Consider electron gas at $T = 0$ and given chemical potential $\mu$, such that $\mu \gg m_e c^2$. In this regime Coulomb interaction is negligible.

a) What is the maximum momentum of an electron in such a gas?

b) What is the density of the electrons at this value of $\mu$.

c) What is the total energy $E$ of such a gas in a volume $V$ containing $N$ electrons. The answer should not contain $\mu$.

d) What is the pressure $P$ of the gas in terms of $\mu$?

e) A nucleus of the substance, called A, can capture an electron and undergo a transformation:

$$A + e^- \rightarrow B + \nu$$

(1)

The mass of nucleus B is larger than the mass of A, therefore the reaction is energetically forbidden under normal conditions. However, at sufficiently high pressure $P > P_{\text{min}}$ the reaction is allowed. Explain why and calculate $P_{\text{min}}$, given the masses $m_A$ and $m_B$. Neglect the masses of electron and neutrino ($m_B - m_A \gg m_e$).